一道三元不等式的“装逼式证明”

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偶然间看到的一道三元不等式，写个“装逼式证明”，待定系数过程留给读者。

题目: 设 $x,y,z\geq 0$，且 $x+2y+3z\leq 6$. 求证:

$xyz(x+y)(y+z)(x+y+z)\leq 27.$

解答（PDF 中已展开）

证明:

令

$X=x+y+z,\qquad Y=y+z,\qquad Z=z.$

则

$X\geq Y\geq Z\geq 0,\qquad X+Y+Z\leq 6,$

且

$P:=xyz(x+y)(y+z)(x+y+z)=XYZ(X-Y)(X-Z)(Y-Z).$

取

$t^3-6t^2+9t-3=0$

的三个正根

$\alpha=4\sin^2 80^\circ,\qquad \beta=4\sin^2 40^\circ,\qquad \gamma=4\sin^2 20^\circ.$

容易验证

$$\begin{cases}\alpha>\beta>\gamma>0,\\ \alpha\beta\gamma=3,\\ (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)=9,\\ \dfrac{1}{\alpha}+\dfrac{1}{\alpha-\beta}+\dfrac{1}{\alpha-\gamma}=1,\\ \dfrac{1}{\beta}+\dfrac{1}{\beta-\alpha}+\dfrac{1}{\beta-\gamma}=1,\\ \dfrac{1}{\gamma}+\dfrac{1}{\gamma-\alpha}+\dfrac{1}{\gamma-\beta}=1.\end{cases}$$

于是，由均值不等式

$$\begin{aligned}
P &= \frac{\sqrt{3}X}{\alpha}\cdot\frac{\sqrt{3}Y}{\beta}\cdot\frac{\sqrt{3}Z}{\gamma}\cdot\frac{\sqrt{3}(X-Y)}{\alpha-\beta}\cdot\frac{\sqrt{3}(X-Z)}{\alpha-\gamma}\cdot\frac{\sqrt{3}(Y-Z)}{\beta-\gamma}\\
&\le\left(\frac{\sqrt{3}}{6}\left(X\left(\frac{1}{\alpha}+\frac{1}{\alpha-\beta}+\frac{1}{\alpha-\gamma}\right)+Y\left(\frac{1}{\beta}+\frac{1}{\beta-\alpha}+\frac{1}{\beta-\gamma}\right)+Z\left(\frac{1}{\gamma}+\frac{1}{\gamma-\alpha}+\frac{1}{\gamma-\beta}\right)\right)\right)^6\\
&=\left(\frac{\sqrt{3}(X+Y+Z)}{6}\right)^6\le(\sqrt{3})^6=27.
\end{aligned}$$

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