数学竞赛每日一练第 8 期解答

2021-06-11 原文链接

网站版本 1c934d989f84 · 文章 ID h_8e6b1ae7797a197d

设 $a_1, a_2, \cdots, x_n \in \mathbb{R}$，$z_1, z_2, \cdots, z_n \in \mathbb{C}$，求证：

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k^2\right) \cdot \left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right).$$

解答（PDF 中已展开）

证明

$\sum\limits_{k=1}^{n} a_k z_k = 0$ 时，结论显然成立.

$\sum\limits_{k=1}^{n} a_k z_k \neq 0$ 时，总能选取 $\theta \in [0, 2\pi]$，使得 $\left(\sum\limits_{k=1}^{n} a_k z_k\right) e^{i\theta}$ 为实数，则用 $z_k e^{i\theta}$ 替换 z_k，不等式不变. 此时设 $z_k = x_k + i y_k (1 \leq k \leq n)$，则

$$\sum_{k=1}^{n} a_k z_k = \sum_{k=1}^{n} a_k x_k$$

于是

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k z_k\right|^2 &= \left(\sum_{k=1}^{n} a_k x_k\right)^2 \\ &\leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} x_k^2\right)\end{aligned}$$

又

$$\begin{aligned}\left|\sum_{k=1}^{n} z_k^2\right| &= \left|\sum_{k=1}^{n}\left(x_k^2 - y_k^2\right) + 2i\sum_{k=1}^{n} x_k y_k\right| \\ &\geq \sum_{k=1}^{n}\left(x_k^2 - y_k^2\right)\end{aligned}$$

则

$$\begin{aligned}&\left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right) \\ \geq &\sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) + \sum_{k=1}^{n}\left(x_k^2 - y_k^2\right) \\ = &2\sum_{k=1}^{n} x_k^2\end{aligned}$$

综上可知，

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k^2\right) \cdot \left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right)$$

成立.