Mathematical Reflections 2023 年第五期征解题

2023-09-28 原文链接

网站版本 1c934d989f84 · 文章 ID h_b2845664c9ffbcfd

1

Mathematical Reflections 2023 年第五期征解题

Junior Problems

程国根 译

J637. 求所有正整数 n，使得

$$3 \cdot n! - 1 = \sqrt{(2n-1)! + 1}.$$

J638. 设 a, b, c 为正实数. 求证:

$$\frac{a+2b}{\sqrt{c(a+2b+3c)}} + \frac{b+2c}{\sqrt{a(b+2c+3a)}} + \frac{c+2a}{\sqrt{b(c+2a+3b)}} \geq \frac{3\sqrt{6}}{2}.$$

J639. 解方程:

$$176x - 4\lfloor x \rfloor^2 - 88\{x\}^2 = 2023,$$

其中 $\lfloor x \rfloor$ 和 $\{x\}$ 分别表示 x 的整数部分和小数部分.

J640. 正实数 x, y, z 满足 $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. 求证:

$$\sqrt{xy+3} + \sqrt{yz+3} + \sqrt{zx+3} \leq \frac{3(x+y+z+1)}{2}.$$

J641. 求所有正整数 a, b，使得

$$\frac{a^2}{b} - \frac{b^2}{a} = 2023.$$

J642. 正实数 a, b, c 满足 $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. 求证:

$$3\sqrt[6]{8abc} \leq \sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}} \leq 3\sqrt{2abc}.$$

Mathematical Reflections 2023 年第五期征解题

Senior Problems

程国根　译

S637. 设 $a,b>0$，$c<0$，且满足

$$2a^2+\frac{1}{8a^2}=7,2b^2+\frac{1}{8b^2}=17,2c^2+\frac{1}{8c^2}=31.$$

求证：

$$(4ab+1)(4bc+1)(4ca+1)=(8abc+1)^2.$$

S638. 设 a,b,c 为正实数．求证：

$$\frac{6a^2}{(2b+c)(2c+b)}+\frac{6b^2}{(2c+a)(2a+c)}+\frac{6c^2}{(2a+b)(2b+a)}\geq 1+\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

S639. 设 $f(n)=\binom{n^2-1}{3},n=2,3,4,\ldots$ 求所有正整数 $k\geq 2$，使得 $f(k+3)=f(k)+2023$．

S640. 设凸四边形 $ABCD$ 有内切圆，圆心为 I，半径为 r．过 I 的一条直线分别和线段 BC,CA,AD 交于点 K,L,M．设 r_1 是与线段 AB,AM,MK 均相切的圆的半径，r_2 是与线段 CD,CK,KM 均相切的圆的半径．记 $a=\dfrac{AL}{LC}$，求证：

$$a\left(\frac{1}{r_1}-\frac{1}{r}\right)=\frac{1}{r_2}-\frac{1}{r}.$$

S641. 求方程的整数解：

$$2x^3-3x^2y^2+2y^3=1.$$

S642. 正实数 a,b,c 满足 $a+b+c=3$．求证：

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\leq\frac{1}{92}\left(68+\frac{1}{abc}\right).$$

Mathematical Reflections 2023 年第五期征解题

Undergraduate Problems

程国根　译

U637. 设 $H_n=\sum\limits_{k=1}^{n}\frac{1}{k}$. 计算:

$$\lim_{n\to\infty} n\left(\frac{n-H_n}{n}\right)^n.$$

U638. 计算:

$$\int\frac{1-\sin x}{3\sin x+5(1+\cos x)e^x}\mathrm{d}x$$

其中 $0<x<\dfrac{\pi}{2}$.

U639. 设 a,b,c 为正实数，$a=\min\{a,b,c\}$ 且 $a^4bc\geq 1$，令

$$F(a,b,c)=\frac{a+b+c}{3}-\sqrt{\frac{ab+bc+ca}{3}}.$$

求证:

$$F(a,b,c)\geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

U640. 数列 $\{x_n\}_{n\geq 2}$ 定义如下：$x_n=\dfrac{\sqrt[n]{e}-1}{\sqrt[n^2]{e}-1}-n$. 求证：$\lim\limits_{n\to\infty}x_n=\dfrac{1}{2}$.

U641. 设 $F(x,y,z)$ 为有理系数多项式.求证:存在有理系数多项式 $Q(x,y,z)$ 和 $R(x,y,z)$，使得

$$P(x,y,z)Q(x,y,z)=R(x^2y,y^2z,z^2x).$$

U642. 计算:

$$\lim_{n\to\infty} n\sin((2\pi n)^p+2^p\pi^p n^{p-1})^{\frac{1}{p}}.$$

Mathematical Reflections 2023 年第五期征解题

Olympiad Problems

程国根　译

O637. 正实数 a,b,c 满足 $a+b+c=ab+bc+ca$. 求证:

$$\sqrt[3]{a^3+7}+\sqrt[3]{b^3+7}+\sqrt[3]{c^3+7}\le 2(a+b+c).$$

O638. 设 $1\le a_1<a_2<\dots$ 为无穷正整数列. 求证: 存在正整数列 $b_1,b_2,\dots$, 满足 $b_i>a_i$, 且满足 $f(b_i+b_j)=f(b_i)+f(b_j)$ 的唯一积性函数 $f:\mathbb{N}\to\mathbb{Z}-\{0\}$ 为 $f(n)=n$.

O639. 正实数 a,b,c,λ 满足

$$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}\le\frac{1}{\lambda}.$$

求证:

$$abc\ge 8\lambda^3 \quad 且 \quad a+b+c+\frac{3abc}{ab+bc+ca}\ge 8\lambda.$$

O640. 数列 $\{a_n\}_{n\ge 0}$ 满足 $a_0>1$, 且对任意 $n\ge 0$, 都有 $a_{n+1}=\dfrac{1+a_n^2}{a}$. 求证:

$$\prod_{k=0}^{n}\frac{1+a_k}{a_k}\ge\left(\frac{(1+a_0)(n+1)}{(1+a_0)n+a_0}\right)^{n+1}.$$

O641. 设数列 $\{F_n\}_{n\ge 0}$ 满足 $F_0=0$, $F_1=1$, 且对任意 $n\ge 0$, 都有 $F_{n+2}=F_{n+1}+F_n$. p,q 为两个不同的 $4k+3$ 型素数. 求所有三元非负整数组 (n,a,b), 使得 $F_n=p^aq^b$.

O642. 点 P 在 $\triangle ABC$ 内部. 设 $D=AP\cap BC$, $E=BP\cap CA$, $F=CP\cap AB$. $\triangle DEF$ 的外接圆和 BC,CA,AB 的另一个交点分别为 D',E',F'. 设 $X=DF'\cap D'F$, $Y=EF'\cap E'F$. 求证: X,P,Y 三点共线.

原文:

Junior Problems

J637. Find all positive integers n such that

$$3 \cdot n! - 1 = \sqrt{(2n-1)! + 1}.$$

Proposed by Adrian Andreescu, Dallas, USA

J638. Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{\sqrt{c(a+2b+3c)}} + \frac{b+2c}{\sqrt{a(b+2c+3a)}} + \frac{c+2a}{\sqrt{b(c+2a+3b)}} \geq \frac{3\sqrt{6}}{2}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J639. Solve the equation

$$176x - 4\lfloor x \rfloor^2 - 88\{x\}^2 = 2023,$$

where $\lfloor x \rfloor$ and $\{x\}$ are the integer part and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J640. Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that

$$\sqrt{xy+3} + \sqrt{yz+3} + \sqrt{zx+3} \leq \frac{3(x+y+z+1)}{2}.$$

Proposed by Marius Stănean, Zalău, România

J641. Find all positive integers a and b such that

$$\frac{a^2}{b} - \frac{b^2}{a} = 2023.$$

Proposed by Mircea Becheanu, Canada

J642. Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove that

$$3\sqrt[6]{8abc} \leq \sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}} \leq 3\sqrt{2abc}.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S637. Let $a, b > 0$ and $c < 0$ such that

$$2a^2 + \frac{1}{8a^2} = 7,\ \ 2b^2 + \frac{1}{8b^2} = 17,\ \ 2c^2 + \frac{1}{8c^2} = 31.$$

Prove that

$$(4ab+1)(4bc+1)(4ca+1) = (8abc+1)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S638. Let a, b, c be positive real numbers. Prove that

$$\frac{6a^2}{(2b+c)(2c+b)} + \frac{6b^2}{(2c+a)(2a+c)} + \frac{6c^2}{(2a+b)(2b+a)} \geq 1 + \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

S639. Let $f(n) = \binom{n^2-1}{3}$, $n = 2, 3, 4, \ldots$. Find all positive integers $k \geq 2$ such that $f(k+3) = f(k) + 2023$.

Proposed by Adrian Andreescu, Dallas, USA

S640. Let $ABCD$ be a convex quadrilateral with incenter I and inradius r. A line passing through I intersects segments BC, CA, AD at points K, L, M, respectively. Let r_1 be the radius of a circle tangent to segments AB, AM, MK and let r_2 be the radius of a circle tangent to segmnets CD, CK, KM. Set $a = AL/LC$. Prove that

$$a\left(\frac{1}{r_1} - \frac{1}{r}\right) = \frac{1}{r_2} - \frac{1}{r}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

S641. Solve in integers the equation

$$2x^3 - 3x^2y^2 + 2y^3 = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S642. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3} \leq \frac{1}{92}\left(68 + \frac{1}{abc}\right).$$

Proposed by Marius Stăncan, Zalău, România

Undergraduate Problems

U637. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Evaluate

$$\lim_{n\to\infty} n\left(\frac{n-H_n}{n}\right)^n.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U638. Evaluate

$$\int \frac{1-\sin x}{3\sin x + 5(1+\cos x)e^x}\,dx$$

where $0 < x < \pi/2$.

Proposed by Mihaela Berindeanu, Bucharest, România

U639. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \geq 1$, and let

$$F(a,b,c) = \frac{a+b+c}{3} - \sqrt{\frac{ab+bc+ca}{3}}.$$

Prove that

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

Proposed by Vasile Cârtoaje, Ploești and Vasile Mircea Popa, Sibiu, România

U640. Let $(x_n)_{n\geq 2}$ be the sequence defined by

$$x_n = \frac{\sqrt[n]{e}-1}{\sqrt[n]{e}-1} - n.$$

Prove that $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Proposed by Dorin Andrica, Cluj-Napoca and Dan-Ștefan Marinescu, Hunedoara, România

U641. Let $P(x,y,z)$ be a polynomial with rational coefficients. Prove that there is a polynomial $Q(x,y,z)$ with rational coefficents such that

$$P(x,y,z)Q(x,y,z) = R\left(x^2y, y^2z, z^2x\right),$$

for some polynomial $R(x,y,z)$ with rational coefficients.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U642. Evaluate

$$\lim_{n\to\infty} n\sin\left((2\pi n)^p + 2^p\pi^p n^{p-1}p\right)^{\frac{1}{p}}$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

Olympiad Problems

O637. Let a, b, c be positive real numbers such that $a+b+c = ab+bc+ca$. Prove that

$$\sqrt[3]{a^3+7} + \sqrt[3]{b^3+7} + \sqrt[3]{c^3+7} \le 2(a+b+c).$$

Proposed by Marius Stănean, Zalău, România

O638. Let $1 \le a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there is a sequence $b_1, b_2, \ldots$ of positive integers with $b_i > a_i$ such that the only multiplicative function $f : \mathbb{N} \longrightarrow \mathbb{Z} - \{0\}$ satisfying the condition $f(b_i + b_j) = f(b_i) + f(b_j)$ is $f(n) = n$ for all n.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O639. Let a, b, c, λ be positive real numbers such that

$$\frac{1}{a+\lambda} + \frac{1}{b+\lambda} + \frac{1}{c+\lambda} \le \frac{1}{\lambda}.$$

Prove that

$$abc \ge 8\lambda^3 \quad \text{and} \quad a+b+c+\frac{3abc}{ab+bc+ca} \ge 8\lambda.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O640. Let $(a_n)_{n\ge 0}$ be the sequence defined by $a_0 > 1$ and $a_{n+1} = \frac{1+a_n^2}{2}$, for all $n \ge 0$. Prove that

$$\prod_{k=0}^{n} \frac{1+a_k}{a_k} \ge \left(\frac{(1+a_0)(n+1)}{(1+a_0)n+a_0}\right)^{n+1}.$$

Proposed by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Italy

O641. Let $(F_n)_{n\ge 0}$, $F_0 = 0$, $F_1 = 1$ be the *Fibonacci* sequence and p, q be distinct primes of the form $4k+3$. Find all triples (n, a, b) of nonnegative integers for which $F_n = p^a q^b$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O642. Point P lies inside triangle ABC. Let $D = AP \cap BC$, $E = BP \cap CA$, $F = CP \cap AB$. The circumcircle of triangle DEF intersects sides BC, CA, AB for the second time at points D', E', F', respectively. Set $X = DF' \cap D'F$, $Y = EF' \cap E'F$. Prove that points X, P, Y are collinear.

Proposed by Waldemar Pompe, Warsaw, Poland

Mathematical Reflections 2023 年第五期征解题

Junior Problems

程国根　译

J637. 求所有正整数 n，使得 $$3\cdot n!-1=\sqrt{(2n-1)!+1}.$$

J638. 设 a,b,c为正实数. 求证：$$\frac{a+2 b}{\sqrt{c(a+2 b+3 c)}}+\frac{b+2 c}{\sqrt{a(b+2 c+3 a)}}+\frac{c+2 a}{\sqrt{b(c+2 a+3 b)}} \geq \frac{3 \sqrt{6}}{2}.$$

J639. 解方程：$$176 x-4\lfloor x\rfloor^2-88\{x\}^2=2023,$$其中 $\lfloor x\rfloor$和 $\{x\}$分别表示 x的整数部分和小数部分.

J640. 正实数 x,y,z满足 $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. 求证：$$\sqrt{x y+3}+\sqrt{y z+3}+\sqrt{z x+3} \leq \frac{3(x+y+z+1)}{2}.$$

J641. 求所有正整数 a,b，使得 $$\frac{a^2}{b}-\frac{b^2}{a}=2023.$$

J642. 正实数 a,b,c满足 $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. 求证：$$3 \sqrt[6]{8 a b c} \leq \sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \leq 3 \sqrt{2 a b c}.$$

Mathematical Reflections 2023 年第五期征解题

Senior Problems

程国根　译

S637. 设 $a,b>0$，$c<0$，且满足 $$2 a^2+\frac{1}{8 a^2}=7,2 b^2+\frac{1}{8 b^2}=17,2 c^2+\frac{1}{8 c^2}=31.$$ 求证：$$(4 a b+1)(4 b c+1)(4 c a+1)=(8 a b c+1)^2.$$

S638. 设 a,b,c为正实数. 求证：$$\frac{6 a^2}{(2 b+c)(2 c+b)}+\frac{6 b^2}{(2 c+a)(2 a+c)}+\frac{6 c^2}{(2 a+b)(2 b+a)} \geq 1+\frac{a^2+b^2+c^2}{a b+b c+c a}.$$

S639. 设 $f(n)=\binom{n^{2}-1}{3}$，$n=2,3,4,\dots$求所有正整数 $k\ge 2$，使得 $f(k+3)=f(k)+2023$.

S640. 设凸四边形 $ABCD$有内切圆，圆心为 I，半径为 r. 过 I的一条直线分别和线段 BC,CA,AD交于点 K,L,M. 设 r_{1}是与线段 AB,AM,MK均相切的圆的半径，r_{2}是与线段 CD,CK,KM均相切的圆的半径. 记 $a=\dfrac{AL}{LC}$，求证：$$a\left(\frac{1}{r_{1}}-\frac{1}{r}\right)=\frac{1}{r_{2}}-\frac{1}{r}.$$

S641. 求方程的整数解：$$2x^{3}-3x^{2}y^{2}+2y^{3}=1.$$

S642. 正实数 a,b,c满足 $a+b+c=3$. 求证：$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3} \leq \frac{1}{92}\left(68+\frac{1}{a b c}\right) .$$

Mathematical Reflections 2023 年第五期征解题

Undergraduate Problems

程国根　译

U637. 设 $H_{n}=\sum\limits_{k=1}^{n}\frac{1}{k}$. 计算：$$\lim_{n\to\infty}n\left(\frac{n-H_{n}}{n}\right)^{n}.$$

U638. 计算：$$\int\frac{1-\sin x}{3\sin x+5(1+\cos x)e^{x}}\mathrm{d}x$$ 其中 $0<x<\dfrac{\pi}{2}$.

U639. 设 a,b,c为正实数，$a=\min\{a,b,c\}$且 $a^{4}bc\ge 1$，令 $$F(a,b,c)=\frac{a+b+c}{3}-\sqrt{\frac{ab+bc+ca}{3}}.$$ 求证：$$F(a,b,c)\ge F\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right).$$

U640. 数列 $\{x_{n}\}_{n\ge 2}$定义如下：$x_{n}=\dfrac{\sqrt[n]{e}-1}{\sqrt[n^{2}]{e}-1}-n$. 求证：$\lim\limits_{n\to\infty}x_{n}=\dfrac{1}{2}$.

U641. 设 $F(x,y,z)$为有理系数多项式. 求证：存在有理系数多项式 $Q(x,y,z)$和 $R(x,y,z)$，使得 $$P(x,y,z)Q(x,y,z)=R(x^{2}y,y^{2}z,z^{2}x).$$

U642. 计算：$$\lim_{n\to\infty}n\sin ((2\pi n)^{p}+2^{p}\pi^{p}n^{p-1})^{\frac{1}{p}}.$$

Mathematical Reflections 2023 年第五期征解题

Olympiad Problems

程国根　译

O637. 正实数 a,b,c满足 $a+b+c=ab+bc+ca$. 求证：$$\sqrt[3]{a^{3}+7}+\sqrt[3]{b^{3}+7}+\sqrt[3]{c^{3}+7}\le 2(a+b+c).$$

O638. 设 $1\le a_{1}<a_{2}<\dots$为无穷正整数列.求证：存在正整数列 $b_{1},b_{2},\dots$，满足 $b_{i}>a_{i}$，且满足 $f(b_{i}+b_{j})=f(b_{i})+f(b_{j})$的唯一积性函数 $f:\mathbb{N}\to\mathbb{Z}-\{0\}$为 $f(n)=n$.

O639. 正实数 a,b,c,λ满足 $$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}\le\frac{1}{\lambda}.$$ 求证：$$abc\ge 8\lambda^{3}\quad\text{且}\quad a+b+c+\frac{3abc}{ab+bc+ca}\ge 8\lambda.$$

O640. 数列 $\{a_{n}\}_{n\ge 0}$满足 $a_{0}>1$，且对任意 $n\ge 0$，都有 $a_{n+1}=\dfrac{1+a_{n}^{2}}{a}$. 求证：$$\prod_{k=0}^{n}\frac{1+a_{k}}{a_{k}}\ge \left(\frac{(1+a_{0})(n+1)}{(1+a_{0})n+a_{0}}\right)^{n+1}.$$

O641. 设数列 $\{F_{n}\}_{n\ge 0}$满足 $F_{0}=0$，$F_{1}=1$，且对任意 $n\ge 0$，都有 $F_{n+2}=F_{n+1}+F_{n}$. p,q为两个不同的 $4k+3$型素数. 求所有三元非负整数组 (n,a,b)，使得 $F_{n}=p^{a}q^{b}$.

O642. 点 P在 $\triangle ABC$内部. 设 $D=AP\cap BC$，$E=BP\cap CA$，$F=CP\cap AB$. $\triangle DEF$的外接圆和 BC,CA,AB的另一个交点分别为 D',E',F'. 设 $X=DF'\cap D'F$，$Y=EF'\cap E'F$. 求证：X,P,Y三点共线.

解答（PDF 中已展开）

```
\section*{Mathematical Reflections 2023 年第五期征解题}
\section*{Junior Problems}
\begin{flushright} 程国根\quad 译\end{flushright}
{\bf J637.} 求所有正整数 $n$，使得 $$3\cdot n!-1=\sqrt{(2n-1)!+1}.$$\vfill
{\bf J638.}设$a,b,c$为正实数. 求证：$$\frac{a+2 b}{\sqrt{c(a+2 b+3 c)}}+\frac{b+2 c}{\sqrt{a(b+2 c+3 a)}}+\frac{c+2 a}{\sqrt{b(c+2
↪  a+3 b)}} \geq \frac{3 \sqrt{6}}{2}.$$\vfill
{\bf J639.}解方程：$$176 x-4\lfloor x\rfloor^2-88\{x\}^2=2023,$$其中$\lfloor x\rfloor$和$\{x\}$分别表示$x$的整数部分和小数部分.
↪  \vfill
{\bf J640.}正实数$x,y,z$满足$x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. 求证：$$\sqrt{x y+3}+\sqrt{y z+3}+\sqrt{z x+3} \leq
↪  \frac{3(x+y+z+1)}{2}.$$\vfill
{\bf J641.} 求所有正整数 $a,b$，使得 $$\frac{a^2}{b}-\frac{b^2}{a}=2023.$$\vfill
{\bf J642.}正实数$a,b,c$满足$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. 求证：$$3 \sqrt[6]{8 a b c} \leq
↪  \sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \leq 3 \sqrt{2 a b c}.$$\vfill\newpage\section*{Mathematical
↪  Reflections 2023年第五期征解题}
\section*{Senior Problems}
\begin{flushright} 程国根\quad 译\end{flushright}
{\bf S637.} 设 $a,b>0$，$c<0$，且满足 $$2 a^2+\frac{1}{8 a^2}=7,2 b^2+\frac{1}{8 b^2}=17,2 c^2+\frac{1}{8 c^2}=31.$$
求证：$$(4 a b+1)(4 b c+1)(4 c a+1)=(8 a b c+1)^2.$$\vfill
{\bf S638.}设$a,b,c$为正实数. 求证：$$\frac{6 a^2}{(2 b+c)(2 c+b)}+\frac{6 b^2}{(2 c+a)(2 a+c)}+\frac{6 c^2}{(2 a+b)(2 b+a)} \geq
↪  1+\frac{a^2+b^2+c^2}{a b+b c+c a}.$$\vfill
{\bf S639.} 设 $f(n)=\binom{n^{2}-1}{3}$，$n=2,3,4,\dots$ 求所有正整数 $k\ge 2$，使得 $f(k+3)=f(k)+2023$. \vfill
```

{\bf S640.}设凸四边形$ABCD$有内切圆，圆心为I，半径为r．过I的一条直线分别和线段BC,CA,AD交于点K,L,M.
↪ 设r_{1}是与线段AB,AM,MK均相切的圆的半径，r_{2}是与线段CD,CK,KM均相切的圆的半径．记$a=\dfrac{AL}{LC}$,
↪ 求证：$$a\left(\frac{1}{r_{1}}-\frac{1}{r}\right)=\frac{1}{r_{2}}-\frac{1}{r}.$$\vfill
{\bf S641.} 求方程的整数解：$$2x^{3}-3x^{2}y^{2}+2y^{3}=1.$$\vfill
{\bf S642.}正实数a,b,c满足$a+b+c=3$．求证：$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3} \leq
↪ \frac{1}{92}\left(68+\frac{1}{a b c}\right) .$$\vfill\newpage\section*{Mathematical Reflections 2023年第五期征解题}
\section*{Undergraduate Problems}
\begin{flushright} 程国根\quad 译\end{flushright}
{\bf U637.} 设 $H_{n}=\sum\limits_{k=1}^{n}\frac{1}{k}$．计算：$$\lim_{n\to\infty}n\left(\frac{n-H_{n}}{n}\right)^{n}.$$\vfill
{\bf U638.} 计算：$$\int\frac{1-\sin x}{3\sin x+5(1+\cos x)e^{x}}\mathrm{d}x$$
其中 $0<x<\dfrac{\pi}{2}$．\vfill
{\bf U639.} 设 a,b,c 为正实数，$a=\min\{a,b,c\}$ 且 $a^{4}bc\ge 1$，令 $$F(a,b,c)=\frac{a+b+c}{3}-\sqrt{\frac{ab+bc+ca}{3}}.$$
求证：$$F(a,b,c)\ge F\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right).$$\vfill
{\bf U640.}数列$\{x_{n}\}_{n\ge 2}$定义如下：$x_{n}=\dfrac{\sqrt[n]{e}-1}{\sqrt[n^{2}]{e}-1}-n$．求证：
↪ $\lim\limits_{n\to\infty}x_{n}=\dfrac{1}{2}$．\vfill
{\bf U641.}设$F(x,y,z)$为有理系数多项式．求证：存在有理系数多项式$Q(x,y,z)$和$R(x,y,z)$，
↪ 使得$$P(x,y,z)Q(x,y,z)=R(x^{2}y,y^{2}z,z^{2}x).$$\vfill
{\bf U642.}计算：$$\lim_{n\to\infty}n\sin ((2\pi n)^{p}+2^{p}\pi^{p}n^{p-1})^{\frac{1}{p}}.$$\vfill\newpage\section*{Mathematical
↪ Reflections 2023年第五期征解题}
\section*{Olympiad Problems}
\begin{flushright} 程国根\quad 译\end{flushright}
{\bf O637.}正实数a,b,c满足$a+b+c=ab+bc+ca$．求证：$$\sqrt[3]{a^{3}+7}+\sqrt[3]{b^{3}+7}+\sqrt[3]{c^{3}+7}\le
↪ 2(a+b+c).$$\vfill
{\bf O638.}设$1\le a_{1}<a_{2}<\dots$为无穷正整数列．求证：存在正整数列$b_{1},b_{2},\dots$，满足$b_{i}>a_{i}$，
↪ 且满足$f(b_{i}+b_{j})=f(b_{i})+f(b_{j})$的唯一积性函数$f:\mathbb{N}\to\mathbb{Z}-\{0\}$为$f(n)=n$．\vfill
{\bf O639.} 正实数 a,b,c,λ 满足 $$\frac{1}{a+\lambda}+\frac{1}{b+\lambda}+\frac{1}{c+\lambda}\le\frac{1}{\lambda}.$$
求证：$$abc\ge 8\lambda^{3}\quad\text{且}\quad a+b+c+\frac{3abc}{ab+bc+ca}\ge 8\lambda.$$\vfill
{\bf O640.}数列$\{a_{n}\}_{n\ge 0}$满足$a_{0}>1$，且对任意$n\ge 0$，都有$a_{n+1}=\dfrac{1+a_{n}^{2}}{a}$．求证：
↪ $$\prod_{k=0}^{n}\frac{1+a_{k}}{a_{k}}\ge \left(\frac{(1+a_{0})(n+1)}{(1+a_{0})n+a_{0}}\right)^{n+1}.$$\vfill
{\bf O641.}设数列$\{F_{n}\}_{n\ge 0}$满足$F_{0}=0$，$F_{1}=1$，且对任意$n\ge 0$，都有$F_{n+2}=F_{n+1}+F_{n}$.
↪ p,q为两个不同的$4k+3$型素数．求所有三元非负整数组(n,a,b)，使得$F_{n}=p^{a}q^{b}$．\vfill
{\bf O642.}点P在$\triangle ABC$内部．设$D=AP\cap BC$，$E=BP\cap CA$，$F=CP\cap AB$．$\triangle
↪ DEF$的外接圆和$BC,CA,AB$的另一个交点分别为$D',E',F'$．设$X=DF'\cap D'F$，$Y=EF'\cap E'F$．求证：
↪ X,P,Y三点共线．\vfill